Introduction to the square root of a 2 by 2 matrix

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The square root of a 2 by 2 matrix $\mathbf{A}$ is another 2 by 2 matrix $\mathbf{B}$ such that $\mathbf{A} = \mathbf{B}^2$, where $\mathbf{B}^2$ stands for the matrix product of $\mathbf{B}$ with itself. We write $\mathbf{A}^{\frac{1}{2}} = \mathbf{B}$. In general, there can be no, two, four or even an infinite number of square root matrices.

For example: $\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix} = \begin{pmatrix}7 & 10\\15 & 22\end{pmatrix}$

If we take $\mathbf{A} = \begin{pmatrix}7 & 10\\15 & 22\end{pmatrix}$, then $\mathbf{A}^{\frac{1}{2}} = \begin{pmatrix}7 & 10\\15 & 22\end{pmatrix}^{\frac{1}{2}} = \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}$

Note that $\mathbf{B} = \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}$ is one of the square roots of $\mathbf{A} = \begin{pmatrix}7 & 10\\15 & 22\end{pmatrix}$.

There may be another root(s) for $\mathbf{A}$.

<u>Questions</u> **(See answers at the end of this file)**

1. Evaluate : $\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$. Hence find **two** square roots of (a) $\begin{pmatrix}2 & 2\\2 & 2\end{pmatrix}$ (b) $\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$.

2. Find **four** values of $\begin{pmatrix}4 & 0\\0 & 9\end{pmatrix}^{\frac{1}{2}}$. Hence find **four** values of $\begin{pmatrix}a & 0\\0 & b\end{pmatrix}^{\frac{1}{2}}$, where $a, b \geq 0$.

3. List all possible matrices $\mathbf{A} = \begin{pmatrix}a & b\\c & d\end{pmatrix}$, where a, b, c and d can be either 0 or 1, such that $\mathbf{A}^{\frac{1}{2}}$ does not exist.

(A) Solving equations method

Suppose we like to find the square root of $\mathbf{A} = \begin{pmatrix}1 & 2\\2 & 5\end{pmatrix}$. So we write:

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}a & b\\c & d\end{pmatrix} = \begin{pmatrix}1 & 2\\2 & 4\end{pmatrix} \Rightarrow \begin{pmatrix}a^2 + bc & b(a+d)\\c(a+d) & cb + d^2\end{pmatrix} = \begin{pmatrix}1 & 2\\2 & 5\end{pmatrix}$$

We get four equations:

$a^2 + bc = 1 \quad \ldots(1)$

$b(a+d) = 2 \quad \ldots(2)$

$c(a+d) = 2 \quad \ldots(3)$

$cb + d^2 = 5 \quad \ldots(4)$

$(2) - (3)$, $(b - c)(a + d) = 0$

Since $a + d \neq 0$ (otherwise (2) or (3) give absurdity), $b - c = 0$, $b = c$.

Hence we get: $a^2 + b^2 = 1 \quad \ldots(5)$

$b(a + d) = 2 \quad \ldots(6)$

$b^2 + d^2 = 5 \quad \ldots(7)$

$(6)^2,\ b^2(a+d)^2 = 4\ ,\ b^2 = \frac{4}{(a+d)^2} \quad \ldots(8)$

$(8) \downarrow (5),\ a^2 + \frac{4}{(a+d)^2} = 1, \quad \ldots(9)$

$(7) - (5), d^2 - a^2 = 4 \quad \ldots(10)$

$d - a = \frac{4}{a+d} \quad \ldots(11) \quad (a + d \neq 0)$

$(11) \downarrow (9), \quad a^2 + \frac{d-a}{a+d} = 1,\ \ a^3 + a^2d + d - a = a + d,\ \ a^3 + a^2d - 2a = 0$

$a(a^2 + ad - 2) = 0$

$\therefore a = 0 \ \text{ or } \ d = \frac{2-a^2}{a} \quad \ldots(12)$

$(12) \downarrow (10),\ a = 0, d = \pm 2 \quad \text{or} \quad a = \pm\frac{1}{\sqrt{2}}, d = \pm\frac{3}{\sqrt{2}}$

The rest can be solved easily, we therefore have:

$$\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}^{\frac{1}{2}} = \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & \pm 2 \end{pmatrix}, \begin{pmatrix} \pm\frac{1}{\sqrt{2}} & \pm\frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} & \pm\frac{3}{\sqrt{2}} \end{pmatrix}$$

3. Prove that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^{\frac{1}{2}}$ does not exist.

List, without prove, all possible matrices $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and d can be either 0 or 1,

such that $\mathbf{A}^{\frac{1}{2}}$ has no real solution(s).

4. Use algebraic method to find $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{\frac{1}{2}}$.

5. Check that the square root of the identity matrix is given by:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{\frac{1}{2}} = \begin{pmatrix} \pm d & \frac{1-d^2}{c} \\ c & \mp d \end{pmatrix}, \begin{pmatrix} \pm d & c \\ \frac{1-d^2}{c} & \mp d \end{pmatrix} \quad ,$$

where $\pm\begin{pmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{pmatrix}, \begin{pmatrix} \pm 1 & 0 \\ c & \mp 1 \end{pmatrix}, \begin{pmatrix} \pm 1 & c \\ 0 & \mp 1 \end{pmatrix}$ are limiting cases.

6. **(a)** Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, find $|\mathbf{A}|$ and $\mathbf{A}^2$. Hence find $\mathbf{A}^{\frac{1}{2}}$.

(b) $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $|\mathbf{A}| = 0$. Given that $\operatorname{tr}(\mathbf{A}) = a + d > 0$.

Show that $\mathbf{A}^2 = \operatorname{tr}(\mathbf{A})\mathbf{A}$. Hence find $\mathbf{A}^{\frac{1}{2}}$.

(B) Diagonalization of Matrrix

Solving equation method in finding the square root of a matrix may not be easy. It involves solving four non-linear equations with four unknowns. You may try this: $\begin{pmatrix} 41 & 12 \\ 12 & 34 \end{pmatrix}^{\frac{1}{2}}$, and soon may give up.

We note that the square root of a **diagonal** matrix can be found easily:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{\frac{1}{2}} = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{b} \end{pmatrix}, \begin{pmatrix} -\sqrt{a} & 0 \\ 0 & \sqrt{b} \end{pmatrix}, \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{b} \end{pmatrix}, \begin{pmatrix} -\sqrt{a} & 0 \\ 0 & -\sqrt{b} \end{pmatrix}.$$

If a matrix is NOT a diagonal matrix, we devise a method called diagonalization to help us.

We proceed with the finding of the eigenvalue(s) and eigenvector(s) of A .

A real number λ is said to be an eigenvalue of a matrix A if there exists a non-zero column vector v such that $Av = \lambda v$ or $(A - \lambda I)v = 0$.

We like to find $\begin{pmatrix} 33 & 24 \\ 48 & 57 \end{pmatrix}^{\frac{1}{2}}$

(1) Eigenvalues

$$A = \begin{pmatrix} 33 & 24 \\ 48 & 57 \end{pmatrix}, \quad v = \begin{pmatrix} x \\ y \end{pmatrix}, \quad (A - \lambda I)v = \begin{pmatrix} 33-\lambda & 24 \\ 48 & 57-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

Now, $(A - \lambda I)v = 0$ has non-zero solution, $|A - \lambda I| = 0$

$$\begin{vmatrix} 33-\lambda & 24 \\ 48 & 57-\lambda \end{vmatrix} = 0$$

$$(33-\lambda)(57-\lambda) - 48 \times 24 = 0$$

$$\lambda^2 - 90\lambda + 729 = 0$$

$$(\lambda - 81)(\lambda - 9) = 0$$

$\therefore \lambda = 9$ or $\lambda = 81$, and these are the **eigenvalues.**

(2) Eigenvectors

We usually would like to find the eigenvector corresponding to each eigenvalue.

The process is called normalization.

For $\lambda_1 = 9$,

$$\begin{pmatrix} 33 & 24 \\ 48 & 57 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 9\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \qquad \begin{cases} 33x_1 + 24y_1 = 9x_1 \\ 48x_1 + 57y_1 = 9y_1 \end{cases}$$

Choose for convenience $24x_1 + 24y_1 = 0 \Leftrightarrow x_1 + y_1 = 0$

$\therefore \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, which is a **eigenvector.**

For $\lambda_2 = 81$,

$$\begin{pmatrix} 33 & 24 \\ 48 & 57 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = 81\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \Leftrightarrow \begin{cases} 33x_2 + 24y_2 = 81x_2 \\ 48x_1 + 57y_1 = 81y_1 \end{cases}$$

Choose $-48x_1 + 24y_1 = 0 \Leftrightarrow 2x_1 - y_1 = 0$

$\therefore \quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, which is another eigenvector.

(3) Diagonalization of matrix

We place two eigenvectors together. Let $\mathbf{P} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$

Consider another matrix:

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 33 & 24 \\ 48 & 57 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 33 & 24 \\ 48 & 57 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 0 \\ 0 & 81 \end{pmatrix}$$

At last we get a matrix **B** which is diagonal with eigenvalues as entries in the main diagonal.

$$\mathbf{B}^{\frac{1}{2}} = \begin{pmatrix} 9 & 0 \\ 0 & 81 \end{pmatrix}^{\frac{1}{2}} = \begin{pmatrix} 3 & 0 \\ 0 & 9 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ 0 & -9 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & -9 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ 0 & 9 \end{pmatrix}$$

Note:

The diagonalization of a matrix may not be a simple subject since $|A - \lambda I| = 0$ may have equal roots or even complex roots. Although most matrices are not diagonal, they can be diagonalized. Not all square matrices can be diagonalised.A thorough study of diagonalization of a matrix is not discussed here.

(4) Finding the square root of the original matrix A

Since $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, we have $\mathbf{A} = \mathbf{P}\mathbf{B}\mathbf{P}^{-1}$

$$\left(\mathbf{P}\mathbf{B}^{\frac{1}{2}}\mathbf{P}^{-1}\right)\left(\mathbf{P}\mathbf{B}^{\frac{1}{2}}\mathbf{P}^{-1}\right) = \mathbf{P}\mathbf{B}^{\frac{1}{2}}(\mathbf{P}^{-1}\mathbf{P})\mathbf{B}^{\frac{1}{2}}\mathbf{P}^{-1} = \mathbf{P}\mathbf{B}^{\frac{1}{2}}\mathbf{B}^{\frac{1}{2}}\mathbf{P}^{-1} = \mathbf{P}\mathbf{B}\mathbf{P}^{-1} = \mathbf{A}$$

So $\left(\mathbf{P}\mathbf{B}^{\frac{1}{2}}\mathbf{P}^{-1}\right)^2 = \mathbf{A}$

(a) $\mathbf{A}^{\frac{1}{2}} = \mathbf{P}\mathbf{B}^{\frac{1}{2}}\mathbf{P}^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 9 \end{pmatrix}\left[\frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\right]$

$$= \begin{pmatrix} 3 & 9 \\ -3 & 18 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -1 & 6 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 4 & 7 \end{pmatrix}$$

(b) $A^{\frac{1}{2}} = PB^{\frac{1}{2}}P^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -3 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -3 & 0 \\ 0 & 9 \end{pmatrix}\left[\frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\right]$

$$= \begin{pmatrix} -3 & 9 \\ 3 & 18 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 8 & 5 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 33 & 24 \\ 48 & 57 \end{pmatrix}^{\frac{1}{2}} = \begin{pmatrix} \mathbf{5} & \mathbf{2} \\ \mathbf{4} & \mathbf{7} \end{pmatrix}, \begin{pmatrix} \mathbf{-5} & \mathbf{-2} \\ \mathbf{-4} & \mathbf{-7} \end{pmatrix}, \begin{pmatrix} \mathbf{1} & \mathbf{4} \\ \mathbf{8} & \mathbf{5} \end{pmatrix}, \begin{pmatrix} \mathbf{-1} & \mathbf{-4} \\ \mathbf{-8} & \mathbf{-5} \end{pmatrix}$$

Questions

7. Real matrix may have **irrational** square roots. Check by **multipication**:

$$\begin{pmatrix}1 & 2\\ 0 & 3\end{pmatrix}^{\frac{1}{2}} = \begin{pmatrix}1 & \sqrt{3}-1\\ 0 & \sqrt{3}\end{pmatrix}, \begin{pmatrix}1 & -\sqrt{3}-1\\ 0 & -\sqrt{3}\end{pmatrix}, \begin{pmatrix}-1 & \sqrt{3}+1\\ 0 & \sqrt{3}\end{pmatrix}, \begin{pmatrix}-1 & -\sqrt{3}+1\\ 0 & -\sqrt{3}\end{pmatrix}$$

8. A simple **real** matrix may have **complex** square roots. Check by **multipication**:

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{\frac{1}{2}} = \frac{1}{2}\begin{pmatrix}1+i & 1-i\\ 1-i & 1+i\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1-i & 1+i\\ 1+i & 1-i\end{pmatrix}, \frac{1}{2}\begin{pmatrix}-1-i & -1+i\\ -1+i & -1-i\end{pmatrix}, \frac{1}{2}\begin{pmatrix}-1+i & -1-i\\ -1-i & -1+i\end{pmatrix}$$

9. A matrix can have both integral and fractional square roots. Use Diagonalization Method to show:

$$\begin{pmatrix}0 & 4\\ -1 & 5\end{pmatrix}^{\frac{1}{2}} = \pm\begin{pmatrix}2 & -4\\ 1 & -3\end{pmatrix}, \pm\begin{pmatrix}\frac{2}{3} & \frac{4}{3}\\ -\frac{1}{3} & \frac{7}{3}\end{pmatrix}$$

Answers

1. $\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}$, Hence:

(a) $\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}^{\frac{1}{2}} = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, \begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix}$

(b) $\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix} = 2\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \Rightarrow \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \left[\pm\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\right]\left[\pm\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\right]$

$$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}^{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}, \begin{pmatrix}-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}$$

2. $\begin{pmatrix}4 & 0\\ 0 & 9\end{pmatrix}^{\frac{1}{2}} = \begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}, \begin{pmatrix}-2 & 0\\ 0 & 3\end{pmatrix}, \begin{pmatrix}2 & 0\\ 0 & -3\end{pmatrix}, \begin{pmatrix}-2 & 0\\ 0 & -3\end{pmatrix}$

$$\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}^{\frac{1}{2}} = \begin{pmatrix}\sqrt{a} & 0\\ 0 & \sqrt{b}\end{pmatrix}, \begin{pmatrix}-\sqrt{a} & 0\\ 0 & \sqrt{b}\end{pmatrix}, \begin{pmatrix}\sqrt{a} & 0\\ 0 & -\sqrt{b}\end{pmatrix}, \begin{pmatrix}-\sqrt{a} & 0\\ 0 & -\sqrt{b}\end{pmatrix}$$

3. $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \Rightarrow \begin{pmatrix}a^2+bc & b(a+d)\\ c(a+d) & cb+d^2\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$

$a^2 + bc = 0 = cb + d^2 \Rightarrow a = \pm d$

(a) Since $b(a+d) = 1, a + d \neq 0,$ and so $a = d \neq 0.$

(b) Lastly, since $c(a+d) = 0 \Rightarrow c = 0$ and so $a^2 + bc = a^2 = 0$, contradicts with (a).

$\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}^{\frac{1}{2}}, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}^{\frac{1}{2}}$ does not exist

$\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}^{\frac{1}{2}} \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^{\frac{1}{2}}$ has no real solution(s).

4. $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix} \Rightarrow \begin{pmatrix}a^2+bc & b(a+d)\\ c(a+d) & cb+d^2\end{pmatrix} = \begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$

We get four equations:

$a^2 + bc = 1 \quad ...(1)$

$b(a + d) = 2 \quad ...(2)$

$c(a + d) = 0 \quad ...(3)$

$cb + d^2 = 1 \quad ...(4)$

From (3), $c = 0 \quad ...(5) \qquad (a + d = 0$ gives contradiction in (2))

From (1), $a^2 = 1, \; a = \pm 1.$

From (3), $d^2 = 1, \; d = \pm 1$

Hence, $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{\frac{1}{2}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$

6. **(a)** $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \quad |\mathbf{A}| = 0 \;, \quad \mathbf{A^2} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 10 & 20 \end{pmatrix} = 5\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = 5\mathbf{A}.$

Hence $\mathbf{A} = \frac{1}{5}\mathbf{A^2} = \left(\frac{1}{\sqrt{5}}\mathbf{A}\right)^2 \;. \quad \mathbf{A}^{\frac{1}{2}} = \frac{1}{\sqrt{5}}\mathbf{A} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{4}{\sqrt{5}} \end{pmatrix}.$

(b) $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad |\mathbf{A}| = ad - bc = 0 \Longrightarrow ad = bc \quad ...(1)$

$\mathbf{A^2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & b(a + d) \\ c(a + d) & cb + d^2 \end{pmatrix} = \begin{pmatrix} a^2 + ad & b(a + d) \\ c(a + d) & ad + d^2 \end{pmatrix}$, by (1).

$= \begin{pmatrix} a(a + d) & b(a + d) \\ c(a + d) & d(a + d) \end{pmatrix} = (a + d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathrm{tr}(\mathbf{A})\mathbf{A}$

$\therefore \mathbf{A} = \frac{\mathbf{1}}{\mathrm{tr}(\mathbf{A})}\mathbf{A^2} = \left(\frac{\mathbf{1}}{\sqrt{\mathrm{tr}(\mathbf{A})}}\mathbf{A}\right)^2$, where $\mathrm{tr}(\mathbf{A}) > 0.$

Hence, $\mathbf{A}^{\frac{1}{2}} = \frac{\mathbf{1}}{\sqrt{\mathrm{tr}(\mathbf{A})}}\mathbf{A}$.

9. Eigenvalues : 4, 1

Eigenvectors: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$\mathbf{P} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}, \mathbf{P^{-1}} = \frac{1}{3}\begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix} \quad , \; \mathbf{B} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$

$\mathbf{B}^{\frac{1}{2}} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}^{\frac{1}{2}} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$

(a) $\mathbf{A}^{\frac{1}{2}} = \mathbf{PB}^{\frac{1}{2}}\mathbf{P^{-1}} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\left[\frac{1}{3}\begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix}\right] = \begin{pmatrix} \frac{2}{3} & \frac{4}{3} \\ -\frac{1}{3} & \frac{7}{3} \end{pmatrix}$

(b) $\mathbf{A}^{\frac{1}{2}} = \mathbf{PB}^{\frac{1}{2}}\mathbf{P^{-1}} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}\left[\frac{1}{3}\begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix}\right] = \begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix}$

$\therefore \begin{pmatrix} 0 & 4 \\ -1 & 5 \end{pmatrix}^{\frac{1}{2}} = \pm\begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix}, \pm\begin{pmatrix} \frac{2}{3} & \frac{4}{3} \\ -\frac{1}{3} & \frac{7}{3} \end{pmatrix}$